I suppose we shall not yet attempt to treat migraine by decompression at the sella Turcica, but I believe it would be efficacious.

POSITIONS OF FOCAL LINES OF A SMALL CYLINDRICAL BEAM OF PARALLEL RAYS TRAVERSING THE CENTRE OF A THIN SYMMETRICAL DOUBLE CONVEX LENS

BY

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The following note was written in the form of a letter to Mr. Percy Bardsley, who was engaged at the time in the preparation of a paper on "A New Form of Bi-focal Lens" that appeared in the March Number of this Journal (pp. 97-101). It was not composed with a view to publication.

WHEN a beam of parallel light of small section is incident upon the centre of a convex lens along the direction of its axis, it converges after traversing the lens to a sharp focus, the *focus* of the lens; the focal length of the lens being defined as the distance of the focus from the unit point of the lens for emergent light. When the thickness of the lens is an inappreciable fraction of the focal length, a condition that may generally be assumed in spectacle lenses, the two unit points practically coincide with the centre of the lens, so that the focal length becomes in this case the distance of the focus from this point.

When, however, the direction of the beam, whether it traverses the centre of the lens or whether it does not, is oblique to the axis, the rays are no longer focussed at a point, but in two short lines (focal lines); one of them, the first focal line, being at right angles to the plane determined by the axes of the beam and lens, while the other, the second focal line, lies in this plane. It will appear that both focal lines are nearer the lens than its focus for axial rays, so that the focal length of the lens is reduced and its power increased for oblique vision. Further, the lengths of the focal lines, and consequently deviation from distinct vision, increases with the obliquity. It is the object of the present note to form an estimate of the extent by which oblique vision through the centre of a lens is affected by these facts. A convex lens has been assumed throughout, but parallel results would have been obtained by the assumption of a concave lens of corresponding power.

Let the plane of Fig. 1 represent that of a plane through the

axis of any lens of any thickness, and let SW and HK be rays of the given beam inclined at angle θ to the axis. Rays of the beam that fall on the upper surface of the lens are refracted downwards and those that fall on the lower surface are refracted upwards. Let SW be the ray that emerges in a direction parallel to its original direction. HK is a neighbouring ray incident on the first surface at K and making an angle θ_1 with CK, the normal, which is the line through K from the centre of curvature of the surface C_1 . On entering the lens the ray is deflected toward the normal, now making an angle θ_2 with it. It traverses the lens,

meeting the further surface at L and making an angle θ_3 with the normal C_2L . On passing into the air it is deflected from this normal making an angle θ_4 with it and meets the ray SWXF₁, at F_p, at an angle θ_5 .

Suppose the figure to rotate through a small angle about C_1C_2 , the axis of the lens; then the plane section of the incident beam describes a solid pencil, while F₁ describes a short line at right angles to the plane of the diagram. Consequently the whole of the beam of which SW may be regarded as the axis is focussed in a short line through F₁ and at right angles to the plane of the diagram. This will be termed the *first focal line*.

The location of F_1 depends on the determination of the angle θ_5 , and this is merely a matter of labour when attacked by the method to be described. The work becomes much simpler if the lens is assumed to be very thin, and the diagram is then reduced to Fig. 2 though it is necessary to refer to Fig. 1 in the determination of the angles.

Let α and β be the angles made by the normal C_1K and C_2L with the axis and μ the index of refraction of the lens.

The relations between the various angles are :

$$\begin{aligned}\theta_1 &= \theta - \alpha \\ \sin \theta_2 &= \frac{1}{\mu} \sin \theta_1 \\ \theta_3 &= \theta_2 + \alpha + \beta \\ \sin \theta_4 &= \mu \sin \theta_3 \\ \theta_5 &= \theta_4 - \beta - \theta_2\end{aligned}$$

These relations enable us to express θ_5 in terms of θ , but the resulting expression is hopelessly cumbersome for computation, and the problem is one of the large class met with in geometrical optics that are most usefully attacked by assuming particular rays and tracing their paths step by step, *i.e.*, by assigning successive values to θ and computing the corresponding values of $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5$.

This has been done for angles of intervals of 5° up to 30° . Since the value of θ_5 depends on the small difference of angles, these must be expressed with great accuracy, and a book of tables that gives trigonometrical functions of angles at intervals of a second of arc has been used throughout.

Taking the special case of a double convex lens formed by two spherical surfaces, each of 100 cms. radius, and assuming a refractive index of 1.5, the values of θ_5 corresponding to different values of θ are :—

$\theta \dots$	10°	15°	20°	25°	30°
$\theta_5 \dots$	$35' 18.4$	$36.46''$	$38' 14.6$	$40' 36.2$	$43.45.9$

θ_5 being found, the determination of F_1 , follows very simply if the lens is assumed thin (see Fig. 2). Let $OK=d$, then, since the sides of a triangle are proportional to the sines of the angles opposite to them :

$$OF_1 = d \cdot \frac{\sin F_1KO}{\sin \theta_5}$$

and since $F_1KO = F_1OT - \theta_5 = 90^\circ - \theta - \theta_5$, $\sin F_1KO = \cos (\theta + \theta_5)$

$$\therefore OF_1 = d \cdot \frac{\cos (\theta + \theta_5)}{\sin \theta_5}$$

The values of OF_1 are given in the second line of the table at the

end of this note. Its value for $\theta=0$ is the focal length of the lens, determined from the relation

$$\frac{1}{f} = (\mu - 1) \left(\frac{1}{r} + \frac{1}{s} \right)$$

d has been taken as 1 cm., and the question might arise whether so large a value would introduce an effect of spherical aberration. Consequently a case has been worked out in which the value of 2 cms. has been assigned to d , but the result proved to be practically unchanged.

The differences between the values of OF_1 and the focal length of 100 cm. are given in line 6, and the figures of a chart in the possession of Mr. Bardsley in line 7. Rather serious differences appear except with the extreme angles.

In all spherical aberration effects, to which those under consideration are closely related, the deviation at first increases with the square of the angle of incidence and subsequently increases more rapidly than the square. To test how far this appears in the present case, line 8 gives the deviations calculated on the assumptions that the value for 30° is correct, and that the deviation from the normal focal length is proportional to the square of the angle of incidence. The law is so closely followed up to 30° that for most purposes it might be used in place of the table.

It is not immediately clear how far the ratio of OF_1 to the focal length would be affected by assigning other values to the radii of curvature and index of refraction, or by supposing the lens to have appreciable thickness. These points have been tested by working out cases of a lens with radii of curvature of 50 cms., a plano-convex lens, a lens with a refractive index of 2.0, and a lens with radii of curvature of 100 cms., but with an exaggerated thickness of 10 cms. In no one of the first three cases does any appreciable change in the ratio appear, but it becomes slightly higher with the thick lens, as might have been anticipated.

But the existence of one focal line necessarily involves that of another.

Consider the rays in the plane through SW and at right angles to the plane of the paper, referred to later on as the inclined plane. Matters seem at first sight complicated by the facts that the sections of the lens surfaces with this plane, though circles, are no longer great circles of the spherical surfaces, and the normals to the surfaces do not lie in the plane; but the solution is not in its general idea difficult.

Imagine a "reference sphere" anywhere in space, and imagine a line drawn through the centre of this reference sphere parallel to the lens axis, and suppose the reference sphere to be observed from a distant point in the direction of this line. In Fig. 3 the

circle is a section of this sphere, Z is its centre, and E is the eye. Let a straight line be drawn through the centre of the reference sphere parallel to the rays of the incident beam. This line is ZK , and the angle at Z is θ . KA is an arc of a great circle on the sphere, and may be taken as the measure of the angle θ . Other arcs on the surface of the sphere presented to E correspond to other angles*—the angles they subtend at the

FIG. 4.

FIG. 5.

centre Z . In Fig. 4 the plane of the paper represents the curved surface of the reference sphere, and lines on it are in reality circles on the reference sphere. The arc AK appears in both Figs. 3 and 4. Now consider a ray of the incident beam that lies in the inclined plane through SW and a little to the left of W as seen from S . This ray meets the surface of the lens to the left of A and a little above it. The normal to the surface is therefore directed upward as viewed from C and to the left. Imagine a line through the centre of the reference sphere parallel to this normal. It will meet the surface at some such point as N in Fig. 4. Now ZN , Z of course cannot be represented in Fig. 4 as it lies vertically below A , is parallel to the normal and ZK is parallel to the incident ray. Therefore the plane of the angle NZK is parallel to the plane of incidence and the angle NZK is the angle of incidence. Calculate the angle of refraction from the relation

$$\text{Sine angle of refraction} = \frac{1}{\mu} \text{Sine angle of incidence}^* \quad (i)$$

and measure off the arc NV such that NZV is the angle of refraction.

* In Figs. 4 and 5 it is simpler to omit the continual reference to Z that follows in the text, understanding by KA , e.g., the angle KZA .

ZV is then parallel to the ray while traversing the lens. It now comes to the further surface. The normal here is directed outwards and upwards toward the left but from the observer, and a line through the centre of the ref. sph. intersects the near surface of the sphere below and to the right of the centre as at S. SZV is the angle of second incidence and its plane is parallel to the plane of this incidence. The angle of refraction into the air is given by the relation

Sine angle of refraction = μ Sine angle of incidence (ii)

and is in the plane of incidence. Set off SR so that the angle SZR is equal to this angle and ZR is the parallel to the course of the ray the other side of the lens. The angle RZK is therefore the angle between the emergent ray and SW, which, no matter whether the lens is thick or thin, goes out in a direction parallel to its direction before incidence. RZK corresponds to θ_2 in the previous determination and its determination is the crux of the problem. So far, the operation applies to any ray falling on any part of the lens, thick or thin, and in general the determination of RK involves very heavy though not difficult operations in spherical trigonometry. But if the lens is very thin, and if the ray traverses the lens very near the centre, the figure squeezes itself up to Fig. 5, N representing the direction of the first normal, which is now only directed upward very slightly and only making a small angle NA with the axis. We have now from spherical trigonometry.

$$\frac{\sin RZK}{\sin NZS} = \frac{\sin VZK}{\sin NZV}$$

NZS is the angle between the normals and is calculable directly if we assume the ray to fall at a known distance, say 1 cm. from the axis; it is, in fact, identical with $\alpha + \beta$ in the former diagram. KZN becomes identical with KZA, the angle of inclination of the beam, and from it NZV is given by equation (i) above, and hence VZK by subtraction. The distance of F₂ the point at which the deviated ray meets the central ray after transmission, is now readily determined. The two rays, both in the inclined plane through SW, make a right angled triangle with line-like section of the thin lens if d , as before, is the distance from the centre at which the ray meets the lens

$$OF_2 = \frac{d}{\sin RZK}$$

Rays above and below the inclined plane meet slightly above or slightly below F₂, as the case may be, and a second focal line is formed roughly at right angles to OF₂, and in the plane of the paper.

Values for OF₂ for angles of 5° up to 30°, have been worked out, and are given in the table.

When the rather curious geometry of the emergent beam is realized, it will be seen that

$$\text{length of first focal line} = \frac{\text{distance between lines}}{\text{OF}_2}$$

$$\text{length of second focal line} = \frac{\text{distance between lines}}{\text{OF}_1}$$

and these lengths, calculated with a slide rule, complete the table.

FOCAL LINES OF A SMALL CYLINDRICAL BEAM OF PARALLEL RAYS PASSING THROUGH THE CENTRE OF A SYMMETRICAL DOUBLE CONVEX LENS OF 100 CM. FOCAL LENGTH.

	ALL MEASUREMENTS ARE IN CM.						
	0°	5°*	10°	15°	20°	25°	30°
1 Inclination of beam to axis of lens							
2 Distance of first focal line from centre of lens	100	98.93	95.71	90.84	84.13	76.31	67.53
3 Length of first focal line for an incident beam of 1 cm. diameter	0	.009	.034	.070	.125	.186	.260
4 Distance of second focal line from centre of lens	100	99.8	99.1	97.7	96.1	93.7	91.2
5 Length of second focal line for an incident beam 1 cm. diameter...	0	.009	.035	.076	.143	.228	.352
6 Shortening of focal length for first focal line	0	1.07	4.29	9.16	15.87	23.69	32.47
7 Figures* from Bardsley's chart ...	0	.70	3.	7.	12.5	21.	31.
8 By law of squares assuming the value for 30 degrees correct	0	.90	3.61	8.12	14.43	22.59	32.47

The figures of the seventh line are from a chart that had been submitted to the writer for criticism, and that gave rise to the investigation.

A CASE OF POISONING BY HOMATROPIN

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CASES of poisoning due to the instillation into the eye of solutions of homatropin, are, I think, sufficiently rare to justify the publication of the following case. As will be seen from the statement, there is room for conjecture as to whether the symptoms were entirely due to idiosyncrasy on the part of the patient, or at least in some degree to impurity of the drug. This was the product