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ASTIGMATIC DIFFERENCE IN REFRACTIVE ERRORS*†

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NOT infrequently the occasion arises to compare and to report on the refractive errors of a patient over a period of time. On such occasions it is often convenient to quote only the change in refraction and, in cases of spherical ametropia, this presents no difficulty. In cases of compound astigmatism, however, the matter is not so simple and the difficulty is usually circumvented by taking one half of the cylindrical component, adding it to the spherical component, and quoting the mean refractive difference between the prescriptions. This procedure has the virtue of simplicity and may be acceptable where the axes of the cylinders in the two prescriptions are approximately the same. There are instances, however, in which the cylinder axes differ considerably—after operation is the obvious example—and it is of importance to ascertain more precisely the actual difference in the refractive condition of the eye as revealed by the original and final prescription.

Suppose for example, that the original prescription was +3·5 D sph., −1·5 D cyl., axis 40°, and that after extraction of the crystalline lens it was +16 D sph., −4 D cyl., axis 90°. If we transpose these prescriptions, it is apparent that the spherical ametropia has increased by ten dioptres (from +2 to +12 D) and that the positive astigmatic error has also increased. The amount of the increase, however, cannot be ascertained by simple subtraction, since the cylinder axes differ by 50°; that is, they are "crossed" obliquely.

It was first pointed out by Stokes (1849) that the resultant of two crossed cylinders can be determined graphically, as in Fig. 1 (opposite), if the dioptric powers C_1 and C_2 are represented by the sides of a parallelogram the diagonal of which represents the cylindrical component C_R of the equivalent sphero-cylindrical lens. The spherical component of this equivalent lens is readily found by resolving the components about a direction perpendicular to C_R. Since meridians of *power* are under consideration, it is necessary to represent the angle between the C_1 and C_2 vectors as 2β, *i.e.* twice the angle β between their axes. Similarly, the direction of the resultant C_R is given by 2α, or twice the angle between the axis of the resultant and that of the component C_1. It is necessary to adopt a convention of signs in order to avoid any ambiguity which might arise in consequence of the change in sign of the trigonometrical ratios of double angles greater than 90°.

Although elegant, this graphical procedure may become tedious if more than a few cases are to be dealt with, and it may become more expedient to use established formulae or, indeed, to consult tables such as those prepared by Emsley and Swaine (1946). Such tables

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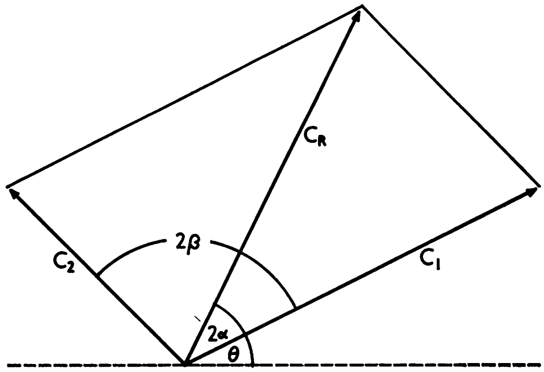
FIG. 1.—Graphical solution for resultant of two obliquely-crossed cylinders.



are impracticable for the present purpose, since they give the resultant of two known components, whereas the effective problem here is to find the second component from a knowledge of the first component and of the resultant. That is to say, expressing the problem in diagrammatic form (Fig. 2), we have to determine the additional, induced, or change in, refractive error (B) when the original refraction (A) and the final refraction (C) are known.

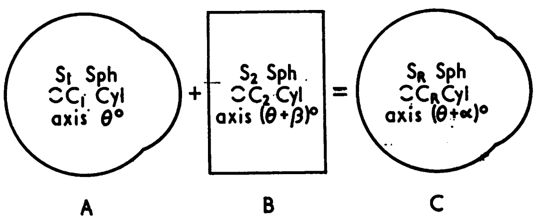


FIG. 2.—Difference in refractive error (B) between original (A) and final (C) prescriptions.

Dealing first with the cylindrical elements only, we find that the sphero-cylindrical lens equivalent to cylindrical lenses C_1 axis $\theta°$ and C_R axis $(\theta + \alpha)°$ is given by Q dioptres sphere, C_2 dioptres cylinder, axis $(\theta + \beta)°$, where

$$C_2 = (C_1^2 + C_R^2 - 2C_1C_R \cos 2\alpha)^{\frac{1}{2}} \quad \text{Equation 1}$$

$$Q = \tfrac{1}{2}(C_1 + C_2 - C_R) \quad \text{Equation 2}$$

and

$$\sin 2\beta = (C_R/C_2) \sin 2\alpha \quad \text{Equation 3}$$

Having established the power and axis of the induced cylindrical component C_2, it remains to determine the additional spherical error S_2 which is given by:

$$S_2 = S_R - S_1 - Q \quad \text{Equation 4}$$

The change in refractive error, S_2 sph. $\bigcirc$ C_2 axis $(\theta + \beta)°$ necessitates, therefore, a calculation in four stages given by the above equations, and it is not suggested that such calculations are worthwhile when the change in either cylinder power or axis is small. They have proved useful, however, when the change was not immediately apparent by inspection.

A series of values of C_2 and β covering likely ranges of cylinder power and axis in the first and final prescriptions has been prepared, therefore, and is given in the appended Table on page 425. Values of S_2 are readily found from Equations 2 and 4 above.

To avoid ambiguity it is necessary to transpose all prescriptions so that the cylinders have a positive sign, and to measure all angles anti-clockwise, converting where necessary those greater than 180° to conventional form. Thus, in the example given, we have $C_1 = +1{\cdot}5$ D axis 130°, $C_R = +4$ D axis 180°, and $\alpha = 50°$, which, by interpolation in the Table, gives $C_2 = +4{\cdot}5$ D and $\beta = 60°$, so that $\theta + \beta = 10°$.

From Equation 2 $\quad Q = \frac{1}{2}(1{\cdot}50 + 4{\cdot}50 - 4{\cdot}00) = +1$ D

and

from Equation 4 $\quad S_2 = (12{\cdot}00 - 2{\cdot}00 - 1{\cdot}00) = +9$ D

The change in prescription, therefore, is +9 D sph., +4·5 D cyl., axis 10°.

This example was chosen deliberately in order to give, at first sight, a somewhat surprising result: namely, that the spherical ametropia has changed by only 9 dioptres instead of the "obvious" 10 dioptres, whereas the astigmatism has changed by 4·5 dioptres, albeit at a different axis. It may be noted that the approximate method of taking half the cylinder would have produced an equivalent result as regards change in "average" dioptric power; that is, a first prescription of (3·50 – 0·75) = 2·75 D sph. and a final prescription of (16·00 – 2·00) = 14 D sph. is a change of 11·25 D, which is equivalent to (9 + 2·25) D sph.

It is hoped that the method outlined above may prove a useful aid to those concerned with assessing accurately the changes in astigmatism as well as in the total refractive error.

I am grateful to Mr. R. Dalgleish of the Department of Ophthalmology, University of Manchester, for indicating the problem, and to my son, J. M. M. Naylor, for his co-operation in compiling the Table.

REFERENCES

EMSLEY, H. H., and SWAINE, W. (1946). "Ophthalmic Lenses", 5th ed., chap. 13, pp. 199–214. Hatton Press, London.

STOKES, G. G. (1849). 19th Meeting of the British Association for the Advancement of Science, 1849. Transactions of the Sections, p. 10. (Publ. 1850).

TABLE

α°	C_R = 1 D, C_1 = 1·0	2·0	3·0	4·0	5·0
10 170	0·4 *51* *129*	1·1 *81* *99*	2·1 *85* *95*	3·1 *87* *93*	4·1 *88* *92*
20 160	0·7 *54* *126*	1·4 *76* *104*	2·3 *82* *98*	3·3 *84* *96*	4·3 *86* *94*
30 150	1·0 *60* *120*	1·7 *75* *105*	2·7 *80* *100*	3·6 *83* *97*	4·6 *85* *95*
40 140	1·3 *65* *115*	2·1 *76* *104*	3·0 *80* *100*	4·0 *83* *97*	4·9 *84* *96*
45 135	1·4 *67* *113*	2·2 *77* *103*	3·2 *81* *99*	4·1 *83* *97*	5·1 *84* *96*
50 130	1·5 *70* *110*	2·4 *79* *101*	3·3 *81* *99*	4·3 *83* *97*	5·3 *85* *95*
60 120	1·7 *75* *105*	2·7 *80* *100*	3·6 *83* *97*	4·6 *84* *96*	5·6 *86* *94*
70 110	1·9 *80* *100*	2·8 *83* *97*	3·8 *85* *95*	4·8 *86* *94*	5·8 *87* *93*
80 100	2·0 *85* *95*	3·0 *87* *93*	4·0 *87* *93*	5·0 *88* *92*	6·0 *88* *92*
90	2·0 *90*	3·0 *90*	4·0 *90*	5·0 *90*	6·0 *90*

α°	C_R = 2 D, C_1 = 1·0	2·0	3·0	4·0	5·0
10 170	1·1 *20* *160*	0·7 *50* *130*	1·3 *75* *105*	2·2 *81* *99*	3·2 *84* *96*
20 160	1·4 *34* *146*	1·4 *55* *125*	2·0 *69* *111*	2·8 *76* *104*	3·7 *80* *100*
30 150	1·7 *45* *135*	2·0 *60* *120*	2·7 *70* *110*	3·5 *75* *105*	4·4 *79* *101*
40 140	2·1 *54* *126*	2·6 *65* *115*	3·3 *72* *108*	4·2 *78* *104*	5·1 *78* *102*
45 135	2·2 *58* *122*	2·8 *68* *112*	3·6 *72* *108*	4·5 *77* *103*	5·4 *79* *101*
50 130	2·4 *62* *118*	3·1 *70* *110*	3·9 *75* *105*	4·8 *78* *102*	5·7 *80* *100*
60 120	2·7 *70* *110*	3·5 *75* *105*	4·4 *78* *102*	5·3 *80* *100*	6·2 *82* *98*
70 110	2·8 *77* *103*	3·8 *80* *100*	4·7 *82* *98*	5·7 *83* *97*	6·7 *85* *95*
80 100	3·0 *84* *96*	3·9 *85* *95*	4·9 *86* *94*	5·9 *86* *94*	6·9 *87* *93*
90	3·0 *90*	4·0 *90*	5·0 *90*	6·0 *90*	7·0 *90*

α°	C_R = 3 D, C_1 = 1·0	2·0	3·0	4·0	5·0
10 170	2·1 *14* *165*	1·3 *26* *154*	1·0 *50* *130*	1·6 *70* *110*	2·4 *77* *103*
20 160	2·3 *28* *158*	2·0 *41* *139*	2·0 *54* *126*	2·6 *65* *115*	3·3 *72* *108*
30 150	2·7 *40* *140*	2·7 *51* *129*	3·0 *60* *120*	3·6 *67* *113*	4·4 *71* *109*
40 140	3·0 *41* *131*	3·3 *58* *122*	3·9 *65* *115*	4·6 *70* *110*	5·4 *73* *107*
45 135	3·2 *54* *126*	3·6 *62* *118*	4·2 *67* *113*	5·0 *72* *108*	5·8 *74* *106*
50 130	3·3 *59* *121*	3·9 *65* *115*	4·6 *70* *110*	5·4 *74* *106*	6·3 *76* *104*
60 120	3·6 *68* *112*	4·4 *72* *108*	5·2 *75* *105*	6·1 *78* *102*	7·0 *79* *101*
70 110	3·8 *75* *105*	4·7 *78* *102*	5·7 *80* *100*	6·6 *82* *98*	7·6 *82* *98*
80 100	4·0 *82* *98*	4·9 *84* *96*	5·9 *85* *95*	6·9 *85* *95*	7·9 *86* *94*
90	4·0 *90*	5·0 *90*	6·0 *90*	7·0 *90*	8·0 *90*

α°	C_R = 4 D, C_1 = 1·0	2·0	3·0	4·0	5·0
10 170	3·1 *13* *167*	2·2 *19* *161*	1·6 *30* *150*	1·4 *50* *130*	1·9 *66* *114*
20 160	3·3 *26* *154*	2·8 *34* *146*	2·6 *45* *135*	2·7 *55* *125*	3·2 *63* *117*
30 150	3·6 *37* *143*	3·5 *45* *135*	3·6 *53* *127*	4·0 *60* *120*	4·6 *66* *114*
40 140	4·0 *43* *137*	4·2 *55* *125*	4·6 *60* *120*	5·1 *65* *115*	5·8 *69* *111*
45 135	4·1 *52* *128*	4·5 *58* *122*	5·0 *63* *117*	5·7 *67* *112*	6·4 *70* *110*
50 130	4·3 *57* *123*	4·8 *62* *118*	5·4 *66* *114*	6·1 *70* *110*	6·9 *72* *108*
60 120	4·6 *65* *115*	5·3 *70* *110*	6·1 *72* *108*	6·9 *75* *105*	7·8 *77* *103*
70 110	4·8 *74* *106*	5·7 *76* *104*	6·6 *78* *102*	7·5 *80* *100*	8·5 *81* *99*
80 100	5·0 *82* *98*	5·9 *83* *97*	6·9 *85* *95*	7·9 *85* *95*	8·9 *86* *94*
90	5·0 *90*	6·0 *90*	7·0 *90*	8·0 *90*	9·0 *90*

α°	C_R = 5 D, C_1 = 1·0	2·0	3·0	4·0	5·0
10 170	4·1 *12* *168*	3·2 *16* *164*	2·4 *23* *157*	1·9 *34* *146*	1·7 *50* *130*
20 160	4·3 *24* *156*	3·7 *30* *150*	3·3 *38* *142*	3·2 *44* *136*	3·4 *55* *125*
30 150	4·6 *35* *145*	4·4 *42* *138*	4·4 *48* *132*	4·6 *54* *126*	5·0 *60* *120*
40 140	4·9 *43* *137*	5·1 *52* *128*	5·4 *57* *123*	5·8 *62* *118*	6·4 *65* *115*
45 135	5·1 *51* *129*	5·4 *56* *124*	5·8 *60* *120*	6·4 *65* *115*	7·1 *67* *113*
50 130	5·3 *55* *125*	5·7 *60* *120*	6·3 *64* *116*	6·9 *67* *113*	7·7 *70* *110*
60 120	5·6 *64* *116*	6·3 *68* *112*	7·0 *71* *109*	7·8 *73* *107*	8·7 *75* *105*
70 110	5·8 *73* *107*	6·7 *75* *105*	7·6 *77* *103*	8·5 *79* *101*	9·4 *80* *100*
80 100	6·0 *82* *98*	6·9 *83* *97*	7·9 *83* *97*	8·9 *84* *96*	9·9 *85* *95*
90	6·0 *90*	7·0 *90*	8·0 *90*	9·0 *90*	10·0 *90*

Top row: Dioptric values of C_R, the cylindrical component of the final prescription.

Second row: Dioptric values of C_1, the cylindrical component of the first prescription.

Extreme left: Values, in degrees, of α, the angle between the axes of the cylindrical components of the first and final prescriptions.

The uppermost (lower-case) figures in each vertical column give the dioptric powers of the induced cylindrical component C_2; the italicised figures give the angle β, in degrees, between the axes of the cylindrical components C_1 and C_2 for corresponding angles of α and its supplement (180-α).